Adaptive Multistep Methods

$$y'(t) = f(t, y), \quad t \in [a, b], \quad y(a) = \alpha \qquad \text{(IVP)}$$

We can take some clues from the adaptive Runge-Kutta techniques when developing an adaptive multistep method. We need an error estimate, and it comes from two computed solutions with known local truncation error forms. In the multistep setting an enticing choice is to use an explicit method for the $\mathrm{O}(h^k)$ method, and an implicit for the $\mathrm{O}(h^{k+1})$ method. These are called *predictor-corrector* methods. We can get both a better solution and a better error estimate by choosing an explicit method of order $\mathrm{O}(h^{k+1})$, but here we will reuse our adaptive RK analysis.

So let's take an Adams-Bashforth (explicit) method with l.t.e $\tau^* = \mathrm{O}(h^k)$:

$$w_{j+1}^* \;=\; w_j + h \sum_{i=0}^{k-1} c_i f(t_{j-i}, w_{j-i}),$$

and an Adams-Moulton (implicit) method with l.t.e. $\tau = \mathrm{O}(h^{k+1})$:

$$w_{j+1} \;=\; w_j + h \left[\hat{c}_{-1} f(t_{j+1}, w_{j+1}) + \sum_{i=0}^{k-1} \hat{c}_i f(t_{j-i}, w_{j-i})\right].$$

The implicit method requires a w_{j+1} on the rhs, and we have a $\mathrm{O}(h^k)$ estimate at our disposal! This gives w_{j+1} explicitly as a *correction* to w_{j+1}^*:

$$w_{j+1} \;=\; w_j + h \left[\hat{c}_{-1} f(t_{j+1}, w_{j+1}^*) + \sum_{i=0}^{k-1} \hat{c}_i f(t_{j-i}, w_{j-i})\right].$$

You recall from the adaptave R-K, that if we want our l.t.e. to be bounded by ϵ, then we can choose a new time step qh, with

$$q = \left(\frac{\epsilon h}{|w_{j+1} - w_{j+1}^*|}\right)^{1/k}.$$

As before, if $q < 1$, then we will need to recompute w_{j+1} and w_{j+1}^*, and if $q > 1$ we may want to increase h. But life is a bit more complicated with multistep methods: the expressions for w_{j+1} and w_{j+1}^* have uniformly spaced t_j. Changing h will have consequences. As usual, we will need an RK method of appropriate l.t.e. to construct an initial history for the multistep methods. But if we change step size, then we may need to reconstruct some local history. For example, if step size is doubled, then we (probably) already have the history we need, but if the step size is halved, what are we to do? Can we employ RK again? Yes. Can we interpolate? Yes. Either way, the complications make us more cautious about changing the step size...