Multistep Methods

$$y'(t) = f(t, y), \quad t \in [a, b], \quad y(a) = \alpha \qquad \text{(IVP)}$$

Single step methods have no memory. Each step is taken as if it were the first step of (IVP) the with initial condition $y(t_k) = w_k$. Can we make use of the information given by $w_0, w_1, \ldots, w_k$ while trying to compute w_{k+1}? We can get m-step "memory" by using the last m of the w_i's. That is, we pretend we know $y(t_i)$ not only at the current time t_k *but also* at $m-1$ previous time steps. Thus a 2-step method uses (t_{k-1}, w_{k-1}) and (t_k, w_k) (instead of only (t_k, w_k)) to determine w_{k+1}. The FTC says

$$y(t+h) - y(t) \;=\; \int_t^{t+h} y'(s)\, ds, \qquad \text{or} \qquad y(t+h) \;=\; y(t) + \int_t^{t+h} f(s, y(s))\, ds,$$

but does not appear to help, for we do not know y' or y in the interval $[t_k, t_{k+1}]$. Multistep methods employ a polynomial interpolator, say P, to approximate f:

$$w_{k+1} \;=\; w_k + \int_{t_k}^{t_{k+1}} P(s)\, ds.$$

The *Adams* class of methods use a Lagrange interpolator and a uniform time step h, and they have l.t.e. equal to 1 plus the degree of P. Specifically, the *Adams-Bashforth* (explicit) methods (AB) fit the data $(t_{k-i}, f(t_{k-i}, w_{k-i}))$, $i = m-1, \ldots, 1, 0$. With h and m fixed, we can integrate P analytically to arrive at values for $c_0, \ldots, c_{m-1}$ in

$$w_{k+1} \;=\; w_k + h \sum_{i=0}^{m-1} c_i f(t_{k-i}, w_{k-i}).$$

For example, the AB two-step (l.t.e. $O(h^2)$) and AB three-step (l.t.e. $O(h^3)$) are

$$w_{k+1} \;=\; w_k + h\, [\tfrac{3}{2} f(t_k, w_k) - \tfrac{1}{2} f(t_{k-1}, w_{k-1})], \text{ and}$$

$$w_{k+1} \;=\; w_k + h\, [\tfrac{23}{12} f(t_k, w_k) - \tfrac{16}{12} f(t_{k-1}, w_{k-1}) + \tfrac{5}{12} f(t_{k-2}, w_{k-2})].$$

The *Adams-Moulton* (implicit) methods (AM) fit the data $(t_{k-i}, f(t_{k-i}, w_{k-i}))$, $i = m-1, \ldots, 1, 0$, *and* $(t_{k+1}, f(t_{k+1}, w_{k+1}))$ to arrive at values for $c_{-1}, c_0, \ldots, c_{m-1}$ in

$$w_{k+1} \;=\; w_k + h\, [c_{-1} f(t_{k+1}, w_{k+1}) + \sum_{i=0}^{m-1} c_i f(t_{k-i}, w_{k-i})].$$

For example, the AM two-step (l.t.e. $O(h^3)$) and AM three-step (l.t.e. $O(h^4)$) are

$$w_{k+1} \;=\; w_k + h\, [\tfrac{5}{12} f(t_{k+1}, w_{k+1}) + \tfrac{8}{12} f(t_k, w_k) - \tfrac{1}{12} f(t_{k-1}, w_{k-1})], \text{ and}$$

$$w_{k+1} \;=\; w_k + h\, [\tfrac{9}{24} f(t_{k+1}, w_{k+1}) + \tfrac{19}{24} f(t_k, w_k) - \tfrac{5}{24} f(t_{k-1}, w_{k-1}) + \tfrac{1}{24} f(t_{k-2}, w_{k-2})].$$

How many *new* function evaluations are required for multistep iterations?