Single-Step Methods and Local Truncation Error

$$y'(t) = f(t,y), \quad t \in [a,b], \quad y(a) = \alpha \qquad \text{(IVP)}$$

Euler's method is an example of a single-step method for (IVP). Generally such a method takes $w_0 = y(a) = \alpha$, and then iterates as

$$w_{k+1} = w_k + h\phi(t_k, w_k),$$

where w_k is our approximation to $y_k = y(t_k)$.

Notice that if we travel h time units from (t_k, y_k) in a straight line of slope m, then we end up at the point $(t_k + h, y_k + hm)$. The value of m that lands us on the solution curve can be found by solving $y(t_k) + hm = y(t_k + h)$ to get the (ideal) slope

$$m = \frac{y(t_k + h) - y(t_k)}{h}.$$

Nevermind that we don't know y_k or y_{k+1}; all of the single-step methods can be viewed as attempts to find this m; in fact, we measure the accuracy of these methods by their *local truncation error* (l.t.e.)

$$\tau(t_k, y_k) = \phi(t_k, y_k) - \frac{y(t_k + h) - y(t_k)}{h}.$$

Euler's method takes $\phi(t_k, y_k) = y'(t_k) = f(t_k, y_k)$, so its l.t.e. is (using Taylor's thm)

$$\tau_{\text{Euler}} = y'(t) - \frac{y(t_k + h) - y(t_k)}{h} = (h/2)y''(\xi) = \mathrm{O}(h).$$

Taylor's theorem suggests higher order methods, e.g.

$$y(t_k + h) = y(t_k) + hy'(t_k) + \frac{h^2}{2}y''(t_k) + \mathrm{O}(h^3)$$

gives the Taylor method of order 2 (with l.t.e. $\mathrm{O}(h^2)$):

$$w_{k+1} = w_k + h\left[f(t_k, w_k) + \frac{h}{2}f'(t_k, w_k)\right].$$

Unfortunately, Euler's method is the only Taylor method that is *general purpose*. The higher order Taylor methods require the evaluation of higher order derivatives, like

$$f'(t_k, w_k) \equiv \frac{d}{dt}f(t,w)|_{(t_k,w_k)} = \frac{\partial f}{\partial t}(t_k, w_k) + f(t_k, w_k)\frac{\partial f}{\partial y}(t_k, w_k).$$

Methods that require the user to provide routines for f' and/or f'', etc. might be very useful in some situations, but are not general purpose...