Euler's Method for the IVP

$$y'(t) = f(t, y), \quad t \in [a, b], \quad y(a) = \alpha \qquad \text{(IVP)}$$

This initial value problem is one of the fundamental problems in scientific computation. Many problems are naturally stated in this form, and many more can be transformed thus if we are willing to view y as a vector-valued function $y : [a, b] \to \mathbb{R}^n$. We have analytic techniques for solving (IVP) in only the simplest cases (e.g. separable first order or linear with constant coefficients, etc.), so approximation techniques are tremendously important.

We will look here at time-stepping methods which, by there very nature, do not even attempt to approximate a solution function y. Instead these methods attempt to compute some points lying close to the graph of y. If a functional form of y is needed, some approximation technique (e.g. Hermite spline interpolation) can be used.

Let D be the vertical strip $[a, b] \times \mathbb{R}$. If $f \in C^0(D)$ and Lipschitz continuous in y on D, then (IVP) has a unique solution which depends continuously on f and α.

The fundamental time-stepping method for (IVP) is Euler's method. It is based on the idea that $f(t, y)$ gives the slope of the tangent line to y at the point (t, y). Then if we step forward along the tangent line h time units, we arrive at the point $(t + h, y(t) + hf(t, y))$, which we hope is close to the point $(t + h, y(t + h))$. An analytic derivation comes from substituting (IVP) into Taylors theorem:

$$y(t + h) \;=\; y(t) + hy'(t) + h^2 y''(\xi)/2 \;=\; y(t) + hf(t, y) + \mathrm{O}(h^2)$$

This process is repeated as t goes from a to b in steps of length h, where $w_j \approx y(a + jh)$:

$w_0 = \alpha; \quad h = (b - a)/N$
For $j = 0, 1, \ldots, N - 1$
    $t = a + jh$
    $w_{j+1} = w_j + hf(t, w_j)$
End (For j)

The truncation errors present in the above algorithm grow from the $\mathrm{O}(h^2)$ Taylor approximation, but when we implement the algorithm in floating point we have rounding errors, too. The following theorem incorporates both types of error. Let $|\delta_j| \leq \delta$ model the rounding errors as

$$u_0 = \alpha + \delta_0, \quad \text{and} \quad u_{j+1} = u_j + hf(a + jh, u_j) + \delta_{j+1},$$

let L be the Lipschitz constant for f in y on D ($|\frac{\partial f}{\partial y}| \leq L$ on D (if it exists)), and $|y''(t) \leq M|$ on $[a, b]$. Then with $t_j = a + jh$

$$|y(t_j) - u_j| \leq \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left(e^{L(t_j - a)} - 1\right) + |\delta_0|e^{L(t_j - a)}.$$

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