Hybrid Methods for Root Finding

We like the certainty of bisection (error bounds and predictable cost) and we like the efficiency of the secant method (superlinear convergence). Can we get both? Yes: the Illinois algorithm, Dekker's method, and Brent's method are some well known general purpose hybrid methods with superlinear convergence. We will talk about Dekker's method and hint at Brent's method, and you may discover one that beats them all.

First let's set up a shill. The *Regula Falsi* method, or *method of false position* is essentially a failed attempt to combine bisection and secant. Having computed approximations $p_0, p_1, \ldots, p_k$ we compute p_{k+1} as follows. As in the secant method, we compute

$$q = p_k - \frac{f(p_k)(p_k - p_{k-1})}{f(p_k) - f(p_{k-1}}.$$

But now, in order to preserve a root-bracketing interval we choose p_{k+1} as follows:

$$\begin{array}{rcl} f(q)f(p_k) > 0 & \rightarrow & p_{k+1} = p_k, \quad p_k = q, \\ f(q)f(p_k) < 0 & \rightarrow & p_{k+1} = q. \end{array}$$

You should try this method by hand for a few problems. If you do, you will see that it is even slower than bisection! The method has a long history, but it really is a poor performer. On the other hand, we can slightly change the definition of q to get a superlinearly convergent method (see, e.g. the Illinois method).

Dekker's method also retains the root bracketing interval of bisection and the superlinear convergence of secant. Dekker avoids the generically slow convergence of false position by sometimes taking the bisection iterate: we compute the secant iterate q as above and the bisection iterate $c = p_{k-1} + (p_k - p_{k-1})/2$. Then

$$\begin{array}{rcl} q \text{ between } c \text{ and } p_k & \rightarrow & p_{k+1} = q, \\ \text{otherwise} & \rightarrow & p_{k+1} = c, \end{array}$$

and to maintain bracketing

$$f(p_{k-1})f(p_{k+1}) < 0 \quad \rightarrow \quad p_k = p_{k-1},$$

and finally, to make p_{k+1} (probably) more accurate than p_k:

$$|f(p_k)| < |f(p_{k+1})| \quad \rightarrow \quad \text{swap } p_k \text{ and } p_{k+1}.$$

Finally note that in Brent's method, if $f(p_{k-2})$, $f(p_{k-1})$ and $f(p_k)$ are pairwise distinct, then we compute q as $q = Q_2(0)$, where Q_2 is the Lagrange interpolator for the data $(f(p_{k-2}), p_{k-2})$, $(f(p_{k-1}), p_{k-1})$ and $(f(p_k), p_k)$. This is called *inverse quadratic interpolation.*