# Hermite Interpolation

Suppose again that we are given a set of knots $(x_j, y_j),\quad j = 0\!:\!n$. We found that with the $x_j$'s distinct (no conditions at all on the $y_j$'s), there is a unique polynomial of degree no more than $n$ that interpolates these points. Here we show that we can control the shape of the interpolator even further.

We would like to have the ability to encode other information in the interpolator. The most common generalization is to include derivative information. We can add to the original interpolation conditions ($P(x_j) = y_j,\quad j = 0\!:\!n$), the derivative conditions

$$P'(x_j) = y'_j,\quad j = 0\!:\!n.$$

Now for each node $x_j$, we need two numbers $y_j$ and $y'_j$. This increase of input data allows us more influence on the interpolator's shape, but requires that we about double its degree: from $(n+1)-1$ to $2(n+1)-1$. For each $j$ the two linear interpolation conditions correspond to two rows

$$\begin{bmatrix} 1 & x_j & x_j^2 & x_j^3 & \dots & x_j^{2n+1} \\ 0 & 1 & 2x_j & 3x_j^2 & \dots & (2n+1)x_j^{2n} \end{bmatrix}$$

of a *generalized Vandermonde system*, $Va = y$, where $a = (a_0, a_1, \dots, a_{2n+1})^T$, $y = (y_0, y'_0, y_1, y'_1, \dots, y_n, y'_n)^T$, and in this standard basis, $P(x) = \sum_{j=0}^{2n+1} a_j x^j$. $V$ is nonsingular iff the $x_j$ are distinct, and thus $P$ is the unique polynomial of degree no more than $2n+1$ which interpolates the data $(x_j, y_j),\ (x_j, y'_j)$; it is called the *Hermite interpolating polynomial.* There are explicit formulas for this polynomial in other bases (see below), and they are simply different representations for the polynomial $P$ above, whose coefficients in the standard basis are $a = V^{-1}y$.

The Lagrange form for the Hermite polynomial takes a very natural form: Define

$$H_{n,i}(x) = [1 - 2(x - x_i)L'_{n,i}(x_i)]L^2_{n,i}(x) \quad \text{and} \quad \hat{H}_{n,i}(x) = (x - x_i)L^2_{n,i}(x),$$

where the $L_{n,i}(x)$ are the standard Lagrange basis polynomials, and check out this handy set of nodal properties:

$$\begin{aligned} H_{n,i}(x_j) &= \delta_{ij}, \quad & \hat{H}_{n,i}(x_j) &= 0, \\ H'_{n,i}(x_j) &= 0, \quad & \hat{H}'_{n,i}(x_j) &= \delta_{ij}. \end{aligned}$$

It is easy to verify now that the "Lagrange form" of this Hermite interpolator is

$$P(x) = \sum_{j=0}^{n} [y_j H_{n,j}(x) + y'_j \hat{H}_{n,j}(x)].$$

If the data are associated with a smooth enough function, then we have an error formula: If $y_j = f(x_j)$, $y'_i = f'(x_j)$ and $[a, b]$ contains the nodes, then $\exists \xi \in (a, b)$ with

$$f(x) = P(x) + \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \prod_{j=0}^{n} (x - x_j)^2.$$