The Cubic Hermite Interpolator

A very useful polynomial is the Hermite cubic, $h(x)$, which interpolates the data (x_0, y_0), (x_1, y_1), (x_0, y_0') and (x_1, y_1'), where we interpret y_0' and y_1' as slopes at x_0 and x_1, respectively. Of course, this is the $n = 1$ case the Hermite interpolator on $n+1$ nodes. It is certainly valid to question the usefulness of an explicit representation of h; nonetheless, let's give it a try. We will define the interpolator, $H(x)$, defined as above but with $x_0 = 0$ and $x_1 = 1$, so that our $h(x)$ is simply

$$h(x) = H\left(\frac{x - x_0}{x_1 - x_0}\right).$$

For an explicit representation of H, we can look at the Lagrange picture or the Vandermonde picture.

The Lagrange picture says we define

$$H_{1i}(x) = [1 - 2(x - x_i)L_{1i}'(x_i)]L_{1i}^2(x) \quad \text{and} \quad \hat{H}_{1i}(x) = (x - x_i)L_{1i}^2(x), \quad i = 0, 1,$$

where $L_{10}(x) = (x - x_1)/(x_0 - x_1)$ and $L_{11}(x) = (x - x_0)/(x_1 - x_0)$ are the linear Lagrange basis functions for (x_0, y_0), (x_1, y_1). For our particular nodes, this simplifies to

$$H_{10}(x) = (1 + 2x)(x - 1)^2, \quad H_{11}(x) = (3 - 2x)x^2, \quad \text{and}$$

$$\hat{H}_{10}(x) = x(x - 1)^2, \quad \hat{H}_{11}(x) = (x - 1)x^2,$$

and our desired polynomial is

$$H(x) = y_0 H_{10}(x) + y_1 H_{11}(x) + y_0' \hat{H}_{10}(x) + y_1' \hat{H}_{11}(x).$$

The Vandermonde picture says we write $H(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, and require

$$\begin{array}{ccccccccc} a_0 & + & a_1 x_0 & + & a_2 x_0^2 & + & a_3 x_0^3 & = & y_0 \\ & & a_1 & + & 2a_2 x_0 & + & 3a_3 x_0^2 & = & y_0' \\ a_0 & + & a_1 x_1 & + & a_2 x_1^2 & + & a_3 x_1^3 & = & y_1 \\ & & a_1 & + & 2a_2 x_1 & + & 3a_3 x_1^2 & = & y_1' \end{array}.$$

For our particular nodes, we solve

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \\ y_1 \\ y_1' \end{bmatrix},$$

and our desired polynomial is

$$H(x) = y_0 + y_0' x + (3(y_1 - y_0) - 2y_0' - y_1')x^2 + (2(y_0 - y_1) + y_0' + y_1')x^3.$$