Gaussian Quadrature

Recall the general quadrature rule

$$\int_a^b f(x)\ dx = \sum_{i=0}^{n} c_i f(x_i) + R,$$

where the *nodes* $x_0 < x_1 < \cdots < x_n \in [a, b]$ are arbitrary, $c_i = \int_a^b L_{ni}(x)\ dx$, $L_{ni}(x)$ is the i^{th} Lagrange basis function for the nodes, and R is the truncation error (see the page *Quadrature*).

Notice that if f is a polynomial of degree n or less, then R is 0, and the rule is *exact*. We say that the *order* of a quadrature rule is the highest degree polynomial for which the rule is exact. Thus the rule above has order at least n.

The *Newton-Cotes* rules use equally space nodes, but if we have freedom to choose our nodes we can do much better:

Suppose there is a polynomial $p_n(x) = c\prod_{i=0}^{n-1}(x - x_i)$ that is orthogonal on $[a, b]$ to $x^i,\ i = 0{:}n{-}1$:

$$\int_a^b x^k p_n(x)\ dx = 0, \quad k = 0{:}n{-}1 \qquad\qquad (\text{O}).$$

Then p_n is orthogonal on $[a, b]$ to any polynomial of degree $\le n{-}1$, and:

If h is any polynomial of degree $2n{-}1$ or less, then we can write

$$h = qp_n + r,$$

where q and r are polynomials of degree at most $n{-}1$, giving

$$\int_a^b h(x)\ dx = \int_a^b q(x)p_n(x)\ dx + \int_a^b r(x)\ dx.$$

But conditions (O) $\Rightarrow \int_a^b q(x)p_n(x)\ dx = 0$, and $p_n(x_i) = 0 \Rightarrow h(x_i) = r(x_i)$, so

$$\int_a^b h(x)\ dx = \int_a^b r(x)\ dx = \sum_{i=0}^{n} c_i r(x_i) = \sum_{i=0}^{n} c_i h(x_i).$$

These conditions on $p_n(x)$ give a *Gaussian Quadrature* (GQ) rule, which is of order at least $2n{-}1$, nearly doubling the order of the Newton-Cotes rules!. Remarkable? You decide. Beautiful? Oh, man! Attainable? Glad you asked:

The orthogonality conditions (O) are a special case of orthogonal polynomials over an interval. For *Legendre* GQ, we map $[a, b]$ to $[-1, 1]$ $(u = 2(x - a)/(b - a) - 1)$, and use the Legendre polynomials which are orthogonal on $[-1, 1]$ (wrt the weight $w(x) \equiv 1$). That is, $p_n(x)$ is (some normalization of) the n^{th} Legendre polynomial (and satisfies (O)). It has the roots we want: Let $g(x) = \prod_{i=1}^{m}(x - x_i)$ be the monic polynomial of degree m with roots consisting of the roots of p_n of odd multiplicity in $(-1, 1)$, then gp_n has roots of even multiplicity, forcing $|\int_{-1}^{1} g(x)p_n(x)\ dx| > 0$. If $m < n$, this integral is 0, so $m \ge n$, and thus p_n must have n distinct roots in $(-1, 1)$. There are other intervals and weight functions which allow GQ for singular functions and infinitely wide intervals, but Legendre GQ is far and away the most often used.