The Fundamental Theorem of Algebra

The fundamental theorem of algebra (FTA) is very easy to state: *If we allow complex numbers, then every polynomial has a root.* It isn't easy to prove (we continue to look for new proofs!). Here are some formulas:

$$ax + b = 0 \longrightarrow x = -b/a$$

$$ax^2 + bx + c = 0 \longrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$ax^3 + bx^2 + cx + d = 0 \longrightarrow x = p + [q + \sqrt{s}]^{1/3} + [q - \sqrt{s}]^{1/3},$$

where $p = -b/(3a)$, $q = p^3 + (bc - 3ad)/(6a^2)$ and $s = q^2 + (c/(3a) - p^2)^3$.

I'm not going to mess with the quartic formula here, but should say that the derivation of the 'depressed cubic' equation (no x^2 term) is due Scipione del Ferro, who didn't publish it, but showed it to his student Antonio Fior as he was dying. Fior then challenged Tartaglia with a list of depressed cubics, and under that pressure Tartaglia independently derived del Ferro's formula. He didn't publish it either, but he showed Cardano the formula with Cardano's promise that he would never publish it. Cardano then got access to del Ferro's notes and found the original formula. Cardano and his student Ferrari then figured out how to transform the general cubic into a depressed cubic, thereby giving the cubic equation(s) above (and the quartic equations (and birthing the first application of complex numbers!)).

Three hundred years later Abel (on the shoulders of Lagrange) showed that *there cannot be* a general quintic, or sextic, etc. equation. So the FTA is an existence theorem, with no hope of a construction using arithmetic and n^{th} roots. Enter numerical analysis. Even though we have formulas for cubic and quartic equations, we typically don't use them: we have numerical methods that are better.

Let's go with the existence theorem: If $p(x) = \sum_{i=0}^{n} a_i x^i$ has a root λ, then $q(x) = p(x)/(x - \lambda)$ is a polynomial of degree $n - 1$, and by the FTA, $q(x)$ has a root. So inductively, we see another way to state the theorem: *any polynomial of degree n can be written as a product of n linear factors:*

$$p(x) = a_n(x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n).$$

How we approximate the λ_i is a very interesting (and evolving) problem in numerical analysis.

Notice that the representation of p as a product of linear divisors is very different than the representations we have seen of p as a linear combination of other polynomials. Here p is uniquely defined as a (scaled) product of linear factors, not a linear combination of basis vectors.

If $z = u + iv$ is complex, then $\overline{z^k} = \overline{(u + iv)^k} = (u - iv)^k = \bar{z}^k$, so if p has real coefficients, and $\lambda = u + iv$ is a root, then $0 = p(\lambda) = \overline{p(\lambda)} = p(\bar{\lambda}) = 0$, and $\bar{\lambda} = u - iv$ is a root of p. Thus, for polynomials with real coefficients, complex roots come in conjugate pairs.