Bisection, Secant and Newton's Methods

We look at three fundamental methods for finding roots of a function $f : \mathbb{R} \to \mathbb{R}$. There are many such methods, some take advantage of the guaranteed smoothness of polynomials and can work with complex roots, some are hybrids (usually combining bisection with other techniques), and so on. For functions of more than one variable the situation is more complicated, but if the function is smooth, most are variations on the Newton/Secant idea.

We've looked in detail at each of the Bisection, Secant, and Newton's methods earlier, so the point here is to compare and contrast.

These methods are to approximate $x^* \in \mathbb{R}$ such that $f(x^*) = 0$. The perspective is that along with other input data, the user will provide a routine which will return $f(x)$, given x. The "Speed" category gives the order of convergence (α and asymptotic error constant (L) under the assumption that the method converges to a simple root. Since Newton's method requires the evaluation of f and f' at each iteration, we also give α_f and L_f, the convergence parameters *per function evaluation* rather than per iteration. This allows a more equitable comparison in the general purpose setting.

In the table, the statement "$x \approx x^*$" means for x near x^*. Of course, this is ambiguous: not only do we not know what "near" means, but we don't know what x^* is, either. So we are basically saying "for x close (or close-enough) to the unknown number x^*".

Method	Bisection	Newton	Secant
Input Req's	• $f \in C^0([a,b])$ • $[a,b] : f(a)f(b) < 0$ • f: $\text{sign}(f(x))$: $x \in [a,b]$	• $f \in C^1(\text{near } x^*)$ • $x_0 \approx x^*$ • f: $f(x)$: $x \approx x^*$ • f': $f'(x)$: $x \approx x^*$	• $f \in C^1(\text{near } x^*)$ • $x_0, x_1 \approx x^*$ • f: $f(x)$: $x \approx x^*$
Speed	• $\alpha = \alpha_f = 1$ • $L = 1/2$	• $\alpha = 2$ • $\alpha_f \approx 1.41$ • $L = \lvert\frac{f''(x^*)}{2f'(x^*)}\rvert$ • $L_f \approx L^{0.41}$	• $\alpha = \alpha_f \approx 1.62$ • $L = L_f \approx \lvert\frac{f''(x^*)}{2f'(x^*)}\rvert^{0.62}$
Convergence	• Yes • Error bound • *a priori* iteration count	• Yes, if $x_0 \approx x^*$ • Error estimate • May not converge • May overflow	• Yes, if $x_0, x_1 \approx x^*$ • Error estimate • May not converge • May overflow