# Shooting Methods

When you have a hammer, everything looks like a nail...

Boundary value problems are generally more difficult to solve than IVP's, but we have a rather mature theory of IVP solvers that we would like to apply to BVP's. The most direct application of IVP methods to BVP's is in *shooting methods*.

Suppose we want to solve the BVP

$$y''(t) = f(t, y, y'), \quad y(a) = \alpha, \quad , y(b) = \beta. \qquad \text{(BVP)}$$

Consider the second order IVP

$$y''(t) = f(t, y, y'), \quad y(a) = \alpha, \quad , y'(a) = x, \quad t \in [a, b],$$

If $f$ is smooth enough, then $y(b)$ is uniquely defined by $x$, and notice that if $x$ is chosen so that $y(b) = \beta$, then we are done.

The game then is to choose $x$, the initial slope of $y$, in such a way as give $y(b) = y(b; \alpha, x) = \beta$. So we define $g(x) = y(b) - \beta$ and try to solve $g(x) = 0$. We have turned our BVP into a root finding problem! We are free to apply any of our root finding techniques to $g$, but notice that one function evaluation of $g$ requires the solution of a second order IVP. That can be found by choosing from among our IVP solvers for the *system*

$$\mathbf{u}' = \mathbf{F}(t, \mathbf{u}), \quad \mathbf{u}(a) = (\alpha, x)^t, \quad t \in [a, b], \text{ where}$$

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \text{ and } \mathbf{F} = \begin{pmatrix} u_2 \\ f(t, u_1, u_2) \end{pmatrix}.$$

To solve $g(x) = 0$, secant is an obvious choice, and for bisection we need to find a bracketing interval of $x$ values. Newton's method requires $g'(x) = \frac{\partial}{\partial x} y(b, \alpha, x)$, and as you might guess, requires the solution of 2 IVP's per iteration. Secant is typically faster than Newton's method, but both require a good initial slope. Inverse interpolation has also been used successfully in shooting methods. To employ that idea, we find an interpolating polynomial (or an interpolating rational function), $P(g)$, for the points $(g(x_i), x_i)$, $\; i = 0, 1, \ldots, n$, and take $x_{n+1} = P(0)$.