Name: ______________________________

(30) 1. Let $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ be real ordered pairs with $x_i \neq x_j$ for $i \neq j$, and $x_i \in [a, b]$, all i.

(a) Give the formulae for the Lagrange basis functions and the Lagrange interpolating polynomial P for this data.

(b) Give the truncation error term for the Lagrange interpolator for this data, assuming that $y_i = f(x_i)$ for a function $f \in C^{n+1}([a, b])$. Include as much information as possible about any parameters not given above.

(c) If Q is a polynomial of degree n that satisfies

$$Q(x_j) = y_j, \quad j = 0, 1, \ldots, n,$$

then explain carefully any relationship between Q and P.

(10) 2. Let $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ be real ordered pairs with $x_0 < x_1 < \cdots < x_n$.

(a) Define the cubic spline interpolator, S, for this data, and describe what conditions it must satisfy.

(b) Are boundary conditions necessary? Why or why not?

(10) 3. Let $f(1) = 1$, $f(2) = 1$ and $f(4) = 2$.

(a) Approximate $f(3)$ using a degree 2 Lagrange interpolator.

(b) If $|f'''(x)| \leq 3$ for $x \in [0, 2\pi]$, give a bound for your error in part (a).

(10) 4. Let $f(.25) = 2$, $f(.5) = 2$, and $f(.75) = 3$.

(a) Approximate $f'(.5)$ using the 3-pt. centered difference formula.

(b) If $|f'''(x)| \leq 3$ for $x \in [0, 1]$, give a bound for your error in part (a).

(10) 5. Write down any difference formula for $f'(x)$ (with its error term), and using that formula describe why numerical differentiation is difficult.

(30) 6. Numerical Integration

(a) Approximate $\int_0^1 6x^5 \, dx$ using Simpson's rule. You do not need to simplify your result.

(b) Describe the adaptive quadrature method in as much detail as you can.

(c) Briefly describe Monte Carlo integration and its relationship with the average value of the function f over a domain Ω.

(d) When is Monte-Carlo integration most often used? Why?