Name: ______________________________

(30) 1. Let $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ be real ordered pairs with $x_i \neq x_j$ for $i \neq j$.

(a) Give the formulae for the Lagrange basis functions, $L_{nk}(x)$, and the Lagrange interpolating polynomial $P(x)$ for this data.

(b) Give the truncation error term for the Lagrange interpolator for this data, assuming that $y_i = f(x_i)$ for a sufficiently smooth function f. Include as much information as possible about any parameters not given above.

(c) The interpolation problem above can also be solved via the Vandermonde equations

$$a_0 + a_1 x_j + a_2 x_j^2 + \cdots + a_n x_j^n = y_j, \quad j = 0, 1, \ldots, n.$$

Explain carefully any relationship between this polynomial, $Q(x) = \sum_{i=0}^{n} a_i x^i$, and the Lagrange interpolating polynomial $P(x)$.

(10) 2. Let $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ be real ordered pairs with $x_0 < x_1 < \cdots < x_n$.

(a) Write down the conditions which define the cubic spline interpolator for this data.

(b) Are other conditions necessary? Explain.

(10) 3. Let $f(0) = 1$, $f(2) = 3$ and $f(3) = 3$. Approximate $f(3)$ using a degree 2 Lagrange interpolator.

(10) 4. Let $f(0.5) = 1$, $f(0.75) = 2$, and $f(1) = 4$. Approximate $f'(0.75)$ using a difference formula.

(10) 5. Write down any difference formula for $f'(x)$ (with its error term), and using that formula describe why numerical differentiation is difficult.

(30) 6. Numerical Integration

(a) i. Approximate $\int_0^1 x^4 \, dx$ using Simpson's rule:
$\int_a^b f(x)dx = \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2)) - \frac{h^5}{90} f^{(4)}(\xi)$.
You do not need to simplify your result.

ii. What does the error term say your error should be?

(b) Describe the adaptive quadrature method in as much detail as you can.

(c) Describe Monte Carlo integration and its relationship with the average value of the function f over a domain Ω.

(d) When is Monte-Carlo integration most often used? Why?