

Linear Combinations II

Let (V, \mathbb{R}) be a vector space and let $v_1, v_2, \dots, v_n \in V$. The *span* of v_1, v_2, \dots, v_n is the set of all vectors that can be made with all possible linear combinations of v_1, v_2, \dots, v_n :

$$\text{Span}(v_1, v_2, \dots, v_n) = \{c_1v_1 + c_2v_2 + \dots + c_nv_n : c_1, \dots, c_n \in \mathbb{R}\}$$

You can create a vector space with this idea since if $v_1, v_2, \dots, v_n \in V$, then $\text{Span}(v_1, \dots, v_n)$ is a vector space, a subspace of V .

$$\text{Span}(v_1, v_2, \dots, v_n) \leq V.$$

To see why, we'll just look at the $n = 2$ case. Let $S = \text{Span}(v_1, v_2)$, with $v_1, v_2 \in V$. Clearly, S is not empty (e.g., $0v_1 + 0v_2$ means $0 \in V$ is also in S). Let $x = c_1v_1 + c_2v_2$ and $y = d_1v_1 + d_2v_2$ be arbitrary vectors in S . Then, checking our closure rules:

$$x + y = (c_1v_1 + c_2v_2) + (d_1v_1 + d_2v_2) = (c_1 + d_1)v_1 + (c_2 + d_2)v_2 \in S,$$

and if α is any scalar

$$\alpha x = \alpha(c_1v_1 + c_2v_2) = (\alpha c_1)v_1 + (\alpha c_2)v_2 \in S.$$

It's like, if V is the entire menu of your restaurant, then $\text{Span}(\text{flour, sugar, eggs, butter})$ is the 'dessert' subspace of that menu, or $\text{Span}(\text{water, coffee})$ the 'beverage' subspace, etc.

We give a name to the situation where $\text{Span}(v_1, v_2, \dots, v_n) = V$ (not just a subspace, but *all* of V):

If $\text{Span}(v_1, \dots, v_n) = V$, then $\{v_1, \dots, v_n\}$ is called a *spanning set* for V

(it is also common to say " v_1, v_2, \dots, v_n spans V "). A spanning set, say $\{v_1, \dots, v_n\}$, is extremely useful since *any* (of infinitely many) vectors in V is a linear combination of just the n vectors v_1, \dots, v_n . In my restaurant, $\text{Span}(\text{water, salt, pepper, coffee, eggs, bacon, flour})$ spans my menu (I can make anything on my menu with these ingredients).

In short, the span of a set of vectors in V is the set of all linear combinations of those vectors, and is a subspace of V ; if the span is all of V , then it is a spanning set for V .

If you were to enlarge your ingredient-vectors for your dessert menu from (flour, sugar, eggs, butter) to (flour, sugar, eggs, butter, milk), then your dessert menu would get bigger, but if you were to enlarge it to (flour, sugar, eggs, butter, eggwhites), it wouldn't (the eggwhites are redundant, since we can get them from the eggs). We can explore this idea mathematically, and for very practical reasons, it will be important.

Again, let $S = \text{Span}(v_1, v_2)$, and let $x \in S$. Then there must be scalars c_1, c_2 so that $x = c_1v_1 + c_2v_2$. Now let $z \in \text{Span}(v_1, v_2, x)$, meaning there there must be scalars d_1, d_2, d_3 so that $z = d_1v_1 + d_2v_2 + d_3x$. But notice that

$$z = d_1v_1 + d_2v_2 + d_3(c_1v_1 + c_2v_2) = (d_1 + d_3c_1)v_1 + (d_2 + d_3c_2)v_2 \in S$$

The vector x (already being in S) is redundant: $\text{Span}(v_1, v_2, x)$ is no different than $\text{Span}(v_1, v_2)$. The term for this redundancy is linear dependence:

If any one (or more) of the vectors v_1, v_2, \dots, v_n is a linear combination of the others, then set $\{v_1, v_2, \dots, v_n\}$ is *linearly dependent*.