Linear Combination

Let $(V, \mathbb{R})$ be a vector space (elements of V are our vectors and numbers in $\mathbb{R}$ are our scalars). Let $u, v \in V$. If $\alpha, \beta \in \mathbb{R}$, the closure rules for a vector space guarantee that

$$\alpha u \in V \quad \text{and} \quad \beta v \in V,$$

and thus

$$\alpha u + \beta v \in V.$$

The vector $w = \alpha u + \beta v$ is called a linear combination of u and v. The linear combination is *the fundamental idea* of all of linear algebra. The scalars in a linear combination are usually called coefficients (or weights), as in "the coefficient of u is α".

We'll get to a more useful definition of linear combination soon, but first a few examples. Let $V = \mathbb{R}^3$ (so V the vector space of 3-tuples with entries in $\mathbb{R}$), and let $u = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$. Then $3u - 2v = \begin{pmatrix} 6 \\ 3 \\ 9 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 7 \end{pmatrix}$ is the linear combination of u and v with 3 the coefficient of u and -2 the coefficient of v. If $w = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$, then $3u - 2v + 5w = (3u - 2v) + 5w = \begin{pmatrix} 16 \\ 10 \\ 27 \end{pmatrix}$ is a linear combination of u, v, and w. So we don't run out of names for vectors and scalars, we index them (name them with subscripts). In the previous example, we could say $a_1 = u, a_2 = v$ and $a_3 = w$, with coefficients $c_1 = 3, c_2 = -2$, and $c_3 = 5$, then

$$3u - 2v + 5w = c_1 a_1 + c_2 a_2 + c_3 a_3 = \sum_{i=1}^{3} c_i a_i = \begin{pmatrix} 16 \\ 10 \\ 27 \end{pmatrix}.$$

In general, if $v_1, v_2, \ldots, v_n \in V$ and $c_1, c_2, \ldots, c_n$ are scalars, then

$$c_1 v_2 + c_2 v_2 + \cdots + c_n v_n = \sum_{i=1}^{n} c_i v_i$$

is a *linear combination* of $v_1, v_2, \ldots, v_n$ with coefficients $c_1, c_2, \ldots, c_n$.

Notice that in our previous example, we could form a matrix A whose j^{th} column is a_j:

$$A = \left[\ a_1, a_2, a_3\ \right] = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -1 & 1 \\ 3 & 1 & 4 \end{bmatrix} \text{ and a vector } c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix},$$

and find that $Ac = a_1 c_1 + a_2 c_2 + a_3 c_3 = c_1 a_1 + c_2 a_2 + c_3 a_3 = \begin{pmatrix} 16 \\ 10 \\ 27 \end{pmatrix}$.

Every matrix-vector product is a linear combination of the columns of the matrix.