

## Linear Combinations III (Linear Independence)

Let  $(V, \mathbb{R})$  be a vector space and let  $v_1, v_2, \dots, v_n \in V$ . We saw that if any of the  $v_i$  is a linear combination of the others, then the set  $\{v_1, v_2, \dots, v_n\}$  is linearly dependent. That is fine and intuitive, but how do we check this? If  $V = \mathbb{R}^n$ , then we can (and will) use systems of equations and matrices, but how?

If  $\{v_1, v_2, \dots, v_n\}$  is *not* linearly dependent, then it is called linearly independent. So any set of vectors in a space  $V$  is either a linearly dependent set or a linearly independent set. We'll lean on the idea of linear dependence to get another definition of linear independence.

Suppose there is some linear combination of  $v_1, v_2, \dots, v_n$  giving  $0 \in V$ :

$$d_1v_1 + d_2v_2 + \dots + d_nv_n = 0.$$

We could always take  $d_1 = d_2 = \dots = d_n = 0$  (the *trivial* combination), but if least one of the  $d_i$  is not 0 (a *nontrivial* combination), then let's say  $d_1 \neq 0$  (by re-labeling, if needed). Then moving the others to the rhs

$$d_1v_1 = -d_2v_2 - d_3v_3 - \dots - d_nv_n$$

and since  $d_1 \neq 0$ :

$$v_1 = -\frac{d_2}{d_1}v_2 - \frac{d_3}{d_1}v_3 - \dots - \frac{d_n}{d_1}v_n = c_2v_2 + \dots + c_nv_n.$$

So, if  $d_k \neq 0$ , then  $v_k$  is in the span of the other  $v_i$ 's and thus  $\{v_1, v_2, \dots, v_n\}$  is linearly dependent.

This gives us a working definition of linear dependence and linear independence:

If there is a nontrivial linear combination of  $v_1, v_2, \dots, v_n$  giving  $0 \in V$ , then  $\{v_1, v_2, \dots, v_n\}$  is linearly dependent.

It's negation:

If the only linear combination of  $v_1, v_2, \dots, v_n$  giving  $0 \in V$  is the trivial combination, then  $\{v_1, v_2, \dots, v_n\}$  is linearly independent.

A popular way of giving the last definition is

$\{v_1, v_2, \dots, v_n\}$  is linearly independent if  $c_1v_1 + c_2v_2 + \dots + c_nv_n = 0$  implies  $c_1 = c_2 = \dots = c_n = 0$ .

Notice that if  $\{v_1, v_2, \dots, v_n\}$  is linearly independent and  $v = c_1v_1 + c_2v_2 + \dots + c_nv_n$  and  $v = d_1v_1 + d_2v_2 + \dots + d_nv_n$ , then  $c_i = d_i$ ,  $i = 1, 2, \dots, n$  (since otherwise  $v - v = 0 \in V$  is a nontrivial linear combination of the  $v_i$ 's giving 0): Independence gives uniqueness.