# $Ax = b$ and Linear Combinations

Suppose our vector space is $V = \mathbb{R}^m$ for some positive integer $m$. Any set of $n$ vectors in $\mathbb{R}^m$, say $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ can be "loaded-in" to an $m \times n$ matrix $A = [a_1, a_2, \ldots, a_n]$. $\mathcal{A}$ is a set (of $n$ vectors), while $A$ is a matrix (the $j^{th}$ column of $A$ is the vector $a_j$, for each $j = 1 : n$).

The solution set of the system of equations $Ax = b$ is the set of all $x \in \mathbb{R}^n$ which satisfies $Ax = b$. Suppose $\hat{x}$ is such a solution. Then, since *every matrix-vector multiplication is a linear combination*,

$$A\hat{x} = b \quad \Longleftrightarrow \quad b = \hat{x}_1 a_1 + \hat{x}_2 a_2 + \cdots + \hat{x}_n a_n.$$

Any solution, $x$, to $Ax = b$ is a vector of coefficients giving $b$ as a linear combination of the column vectors of $A$. This means

$Ax = b$ has a solution if and only if $b \in \text{Span}(a_1, a_2, \ldots, a_n)$.

Does $Ax = b$ have more than one solution? Suppose $A\hat{x} = b$ and $A\hat{y} = b$ with $\hat{x} \neq \hat{y}$. Then $A\hat{x} = A\hat{y}$. Or $A\hat{x} - A\hat{y} = 0$. Or $A(\hat{x} - \hat{y}) = 0$. But the vector $\hat{x} - \hat{y} \neq 0$, so the columns of $A$ must be linearly dependent. Now suppose the columns of $A$ are linearly dependent. Then there is a nonzero vector $z$ with $Az = 0$, and if $Ax = b$, then $A(x + z) = Ax + Az = b + 0 = b$, so there is more than one solution ($x$ and $x + z$).

A consistent system $Ax = b$ has exactly one solution if and only if
the columns of $A$ are linearly independent.

Putting these ideas together we can say that

$Ax = b$ has exactly one solution for *any* $b \in \mathbb{R}^m$ if and only if
$\text{span}(a_1, a_2, \ldots, a_n) = \mathbb{R}^m$ *and* $\{a_1, a_2, \ldots, a_n\}$ is linearly independent.

Thus, by combining the ideas of spanning set and linearly independent set into the idea of a basis, we can say

$Ax = b$ has exactly one solution for *any* $b \in \mathbb{R}^m$ if and only if
the columns of $A$ form a basis for $\mathbb{R}^m$.

The idea of spanning set gives us the existence of a solution (consistency of the system), and the idea of linear independence gives us uniqueness of the solution.