Dimension

So, a *basis* for a vector space V is a set of vectors in V which (1) is a spanning set for V, and (2) is linearly independent.

The following theorem may seem technical, but it gives a vital connection between spanning and linear independence, the two properties of a basis.

Theorem: Let $\mathcal{S}$ be a spanning set of n vectors for a v.s. V. Then any set of more than n vectors in V is linearly dependent.

We'll **prove this in the case** $V \leq \mathbb{R}^k$: Let $\mathcal{S} = \{a_1, a_2, \ldots, a_n\}$ be a spanning set for V, and create the matrix $A = [a_1, a_2, \ldots, a_n]$. Now do the same thing for any set $\{b_1, b_2, \ldots, b_m\}$ of vectors in V: $B = [b_1, b_2, \ldots, b_m]$. Clearly, A is $k \times n$ and B is $k \times m$.

Since $b_i \in V$ and $\mathcal{S}$ spans (all of) V, we know b_i is some linear combination of $\{a_1, a_2, \ldots, a_n\}$:

$$b_i = c_{1i}a_i + c_{2i}a_2 + \cdots + c_{ni}a_n, \quad i = 1 : m$$

(so $c_i = (c_{1i}, c_{2i}, \ldots, c_{ni})^T$ is the vector of coefficients for making b_i as a linear combination of the a_j's). We can write these as matrix-vector products:

$$b_i = Ac_i, \quad i = 1 : m,$$

which says the B matrix we introduced earlier can be written as:

$$B = [Ac_1, Ac_2, \ldots, Ac_m].$$

As above, we can create the matrix C by loading in the coefficient columns: $C = [c_1, c_2, \ldots, c_m]$, and this whole proof so far can be summarized as

$$B = AC.$$

How many rows and columns does C have? Each column (c_i) has n elements, and there are m columns (one for each b_i), so C is $n \times m$. If $m > n$, then the columns of C are linearly dependent (there must be a free variable). But that means there is a nonzero vector y giving $Cy = 0$. But that means $By = ACy = 0$. And that means (definition of linear independence) the columns of B are linearly dependent. $\square$

Theorem: If $\mathcal{S}$ and $\mathcal{B}$ are bases for a v.s. V, then they must each contain the same number of vectors.
Proof: $\mathcal{B}$ is linearly independent, so it can't have more vectors than $\mathcal{S}$ (since $\mathcal{S}$ is a spanning set). Likewise $\mathcal{S}$ is linearly independent, so it can't have more vectors than $\mathcal{B}$ (since $\mathcal{B}$ is a spanning set). (i.e. $\{m \leq n$ and $n \leq m\} \Longrightarrow m = n$.) $\square$

Definition: The *dimension* of a v.s. V is the number of vectors in a basis for V.