

Vector Coordinates

So we have a vector space, V , with a basis $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$. Since \mathcal{B} is a spanning set for V , every vector in V is a linear combination of the basis vectors, and since \mathcal{B} is a linearly independent set, the coefficients (scalars) in any such linear combination are unique.

If you take any $v \in V$, then there is a unique vector of coefficients,

$$(c_1, c_2, \dots, c_n)^T \in \mathbb{R}^n \quad \text{so that} \quad v = c_1v_1 + c_2v_2 + \dots + c_nv_n.$$

To each $v \in V$ there's exactly 1 such $c \in \mathbb{R}^n$ & to each $c \in \mathbb{R}^n$ there's exactly 1 such $v \in V$.

The vector c is called the *coordinate vector of v with respect to \mathcal{B}* . Since this coordinate vector depends on both v and \mathcal{B} , we use the notation $[v]_{\mathcal{B}}$, so every vector, $v \in V$, is associated uniquely to the vector $[v]_{\mathcal{B}} \in \mathbb{R}^n$.

So, if V is any v.s. with basis $\mathcal{F} = \{f_1, f_2, f_3\}$, and v is any vector in V , then $[v]_{\mathcal{F}} \in \mathbb{R}^3$.

$$\text{If } v = 2f_1 - 3f_2 + f_3, \text{ then } [v]_{\mathcal{F}} = (2, -3, 1)^T,$$

if $w = -f_1 + 2f_2 + 4f_3$, then $[w]_{\mathcal{F}} = (-1, 2, 4)^T$, etc.

In an arbitrary v.s. finding $[v]_{\mathcal{F}}$ may be difficult, but if $V = \mathbb{R}^n$ for some n , then finding $[v]_{\mathcal{F}}$ is (at least conceptually) very simple. If $x \in \mathbb{R}^n$, then

$$x = (x_1, x_2, \dots, x_n)^T = x_1e_1 + x_2e_2 + \dots + x_ne_n, \quad \text{so} \quad [x]_{SOB} = x$$

(make sure this makes sense). If $\mathcal{F} = \{f_1, f_2, \dots, f_n\}$, and if $x = c_1f_1 + c_2f_2 + \dots + c_nf_n$, then $[x]_{\mathcal{F}} = (c_1, c_2, \dots, c_n)^T$. Now (because $\mathcal{F} \subset \mathbb{R}^n$, we can) form the matrix $F = [f_1, f_2, \dots, f_n]$ by setting its j^{th} column to f_j . Evidently, $x = [x]_{SOB} = Fc$, and we can find $c = [x]_{\mathcal{F}}$ by solving the (nonsingular) system $F[x]_{\mathcal{F}} = [x]_{SOB}$.

F is the matrix for \mathcal{F} , and if we multiply $[x]_{\mathcal{F}}$ (on the left) by F , then we get $[x]_{SOB}$, so F is also called the *transition matrix* from \mathcal{F} to SOB. Multiplication by F is the function that takes coordinates wrt \mathcal{F} to coordinates wrt SOB. We can solve $F[x]_{\mathcal{F}} = [x]_{SOB}$ for $[x]_{\mathcal{F}}$: $[x]_{SOB} = F^{-1}[x]_{\mathcal{F}}$, so F^{-1} must be the transition matrix from SOB to \mathcal{F} .

Now let's figure out how to go from a basis \mathcal{F} to a basis \mathcal{G} . As usual, if $\mathcal{G} = \{g_1, g_2, \dots, g_n\}$, we let $G = [g_1, g_2, \dots, g_n]$. Our work above says: for any $x \in \mathbb{R}^n$ $G[x]_{\mathcal{G}} = [x]_{SOB} = F[x]_{\mathcal{F}}$; or simply

$$G[x]_{\mathcal{G}} = F[x]_{\mathcal{F}}.$$

This is definitely worth remembering! Solving this for $[x]_{\mathcal{F}}$ tells us that the transition matrix from \mathcal{G} to \mathcal{F} is $F^{-1}G$ (and, solving for $[x]_{\mathcal{G}}$, we see that the transition matrix from \mathcal{G} to \mathcal{F} is $G^{-1}F$).

Let $V = \mathbb{R}^2$ and let $\mathcal{F} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ be a basis for V . Then $F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is the matrix of \mathcal{F} and is the transition matrix from \mathcal{F} to SOB. Likewise, if $\mathcal{G} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$, then $G = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ is both the matrix of \mathcal{G} and the transition matrix from \mathcal{G} to SOB. The transition from \mathcal{F} to \mathcal{G} is $G^{-1}F$, while the transition from \mathcal{G} to \mathcal{F} is $F^{-1}G = (G^{-1}F)^{-1}$.