Linear Combinations IV (Basis)

Let $(V, \mathbb{R})$ be a vector space and let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\} \subseteq V$. If *every* $v \in V$ is a linear combination of vectors in $\mathcal{B}$, then we say that $\mathcal{B}$ spans V. If the only way to get $0 \in V$ as a linear combination of vectors in $\mathcal{B}$ is with all coefficients equal 0, then we say $\mathcal{B}$ is a linearly independent set in V. If we combine these two ideas, then we have a *basis* for V.

A *basis* for a vector space V is a set of vectors in V which (1) spans V and (2) is linearly independent.

This means that for *any* $v \in V$, there exist unique scalars $c_1, c_2, \ldots, c_n$ so that

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \sum_{i=1}^{n} c_i v_i.$$

The existence of such $c_i's$ comes from B being a spanning set for V.

The uniqueness of the $c_i's$ comes from B being linearly independent:
$v = \sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} d_i v_i$, then $0 = \sum_{i=1}^{n} (c_i - d_i) v_i$, so ($\mathcal{B}$ lin. ind.) $c_i = d_i, \; i = 1:n$.

Suppose our v.s. is $V = \mathbb{R}^2$. Then a basis for V is $\mathcal{B} = \{e_1, e_2\}$. How do you know? Each vector in V can be written in exactly one way as a linear combination of the vectors in $\mathcal{B}$. Notice that these 2 vectors – together with all of their linear combinations – give us all of V.

Suppose our v.s. is $V = \mathbb{R}^2$. Then a basis for V is $\mathcal{B} = \{(1,1)^T, (1,-1)^T\}$. How do you know? Each vector in V can be written in exactly one way as a linear combination of the vectors in $\mathcal{B}$. Notice that these 2 vectors – together with all of their linear combinations – give us all of V.

How many different bases are there for $\mathbb{R}^2$?

Suppose our v.s. is $V = \mathbb{R}^3$. Then a basis for V is $\mathcal{B} = \{e_1, e_2, e_3\}$. How do you know? Each vector in V can be written in exactly one way as a linear combination of the vectors in $\mathcal{B}$. Notice that these 3 vectors – together with all of their linear combinations – give us all of V.

Suppose our v.s. is $V = \mathcal{P}_3$. Then a basis for V is $\mathcal{B} = \{1, x, x^2\}$. How do you know? Each vector in V can be written in exactly one way as a linear combination of the vectors in $\mathcal{B}$. Notice that these 3 vectors – together with all of their linear combinations – give us all of V.

Suppose our v.s. is $V = \mathbb{R}^{2\times 2}$. Then a basis for V is $\mathcal{B} = \{e_1 e_1^T, e_1 e_2^T, e_2 e_1^T, e_2 e_2^T\}$. How do you know? Each vector in V can be written in exactly one way as a linear combination of the vectors in $\mathcal{B}$. Notice that these 4 vectors – together with all of their linear combinations – give us all of V.

In all of these cases, we have a small set of vectors (the basis), which when combined with their linear combinations, give the whole vector space.